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The construction of some important classes of generalized coherent states: the nonlinear coherent states method

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Abstract
Considering some important classes of generalized coherent states known in the literature, we demonstrated that all of them can be created via conventional methods, i.e. the 'lowering operator eigenstate' and the 'displacement operator' techniques using the 'nonlinear coherent states' approach. As a result we obtained a 'unified method' to construct a large class of coherent states which have already been introduced by different prescriptions.

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1. Introduction

Coherent states (CSs) are venerable objects in physics with many applications in most of the fields of physics and mathematical physics (Klauder and Skagerstam 1985, Ali *et al* 2000) from solid states physics to cosmology, and occurring in the core of quantum optics. Along with a generalization of coherent states J-P Gazeau and J R Klauder recently proposed new coherent states (GK) for systems with discrete or continuous spectra, which are parametrized by two real and continuous parameters $J \geqslant 0$ and $-\infty < \gamma < \infty$, denoted by $|J, \gamma\rangle$ (Gazeau 1999). They involve the representation of the discrete series $su(1, 1)$ Lie algebra, Coulomb-like spectrum, Pöschl–Teller and the infinite square well potentials (Antoine *et al* 2001).

More recently Klauder, Penson and Sixdeniers introduced another important class of generalized CSs through solutions of the Stieltjes and Hausdorff moment problem (Klauder 2001). The constructed CSs in their paper which we will denote by $|z\rangle_{\mathrm{KPS}}$ have these properties: (i) *normalization* (ii) *continuity in the label* and (iii) they form an *overcomplete* set which allows a resolution of unity with a positive weight function. Then they have studied the statistical properties of these states by calculating Mandel parameter, and the geometry of these states by a metric factor, both analytically and numerically. Finally they used some estimations to find the proper Hamiltonian for some sets of their CSs.

The generalized CSs mentioned above were neither defined as eigenstates of an annihilation operator, known as Barut–Girardello (BG) CSs (Barut and Girardello 1971), nor resulting from the action of a displacement operator on a reference state, say vacuum state, known frequently as Gilmore–Perelomov (GP) CSs (Gilmore 1972, Perelomov 1972). In other words the algebraic and symmetry considerations of the above two sets of states have not been clear, there. The same situation holds for the generalized CSs proposed by Penson and Solomon; we denote them by $|z, q\rangle_{\mathrm{PS}}$ (Penson and Solomon 1999). Due to these, our aim is to establish these subjects and also to find a simpler way to relate the constructed CSs to an exact expression of the Hamiltonian, out of any estimation. In a sense our present paper may be considered as the completion of the works of Gazeau *et al* (1999), Klauder *et al* (2001) and Penson and Solomon (1999).

Our procedure is based on the conjecture that these states may be studied in the so-called *nonlinear* (NL) *coherent states* or f*-deformed coherent states* category (Man'ko *et al* 1997). This is the main feature of our work. Nonlinear CSs have attracted much attention in recent years, mostly because they exhibit nonclassical properties. As we know, up to now many quantum optical states such as q-deformed CS (Man'ko *et al* 1997), negative binomial state (Wang 1999, Wang and Fu 1999), photon added (and subtracted) CS (Sivakumar 2000, 1999, Naderi *et al* 2004), the centre of mass motion of a trapped ion (de Matos Filho and Vogel 1996), some nonlinear phenomena such as a hypothetical *'frequency blue shift'* in high intensity photon beams (Man'ko and Tino 1995) and recently after proposing f-bounded CS (Re'camier and Ja'uregui 2003), the binomial state (or displaced excited CS) (Roknizadeh and Tavassoly 2004) have been considered as some sort of nonlinear CSs.

We attempt now to demonstrate that all sets of KPS coherent states, including PS coherent state and the discrete series representations of the group $SU(1, 1)$ (BG and GP coherent states) and $su(1, 1)$-Barut–Girardello CSs for Landau-level (LL) (Fakhri 2004) can be classified in the nonlinear CSs with some special types of nonlinearity function $f(n)$, by which we may obtain the 'deformed annihilation and creation operators', 'generalized displacement operator' and the 'dynamical Hamiltonian' of the system. Based on these results it will be possible to reproduce all of the above CSs through the conventional fashion, i.e. by annihilation and displacement operator definition. In a general view the formalism presented in this paper provides a *unified* approach to construct all the employed CSs already introduced in different ways. In another direction as a matter of fact introducing the ladder operators related to the above CSs may be regarded as a first step in the process of generation of these states in some experimental realization schemes in quantum optics, when one intended to perform the interaction Hamiltonian describes formally the interaction between atoms and electromagnetic field.

In addition to these, three interesting and new remarkable points, which seemingly have not been pointed out so far, emerge from our studies. The first is that we can construct a new family of CSs (named the *dual* set), other than the KPS and PS coherent states (Ali *et al* 2004, Roy and Roy 2000). The second is that the Hamiltonian proposed in Man'ko *et al* (1997) and in other works that cited him (see, e.g., Sivakumar (2000)) must be reformed in view of the *action identity* requirement imposed on the generalized CSs of Klauder (1998). We should quote here that recently some authors (for instance see El Kinani and Daoud 2003) have used the normal ordering form (factorization) for their Hamiltonians. But they sometimes made this suggestion for drastic simplification, without any deep physical basis (Speliotopoulos 2000). Indeed they used the *supersymmetric quantum mechanics* (SUSQM) techniques as a *'mathematical tool'* to find the ladder operators for their Hamiltonians. Interestingly our formalism for solvable Hamiltonians gives an easier and clearer method to obtain these operators whenever necessary. We will pay more attention to this result in the conclusion. Thirdly, our results give us the opportunity to observe that for some sets of the KPS coherent states, the nonlinearity phenomena will be visible in 'low intensities of light', a fact that had been hidden, until we discovered the nonlinear nature of them explicitly. This would be important experimentally, since as Man'ko *et al* experienced, the nonlinear phenomena in q-oscillators can be detected only in high intensity photon beams (Man'ko and Tino 1995). Therefore if anyone can generate these particular sets of KPS coherent states by interaction of a field and atoms, it will be easier to detect these phenomena.

The plan of this paper is as follows: for the sake of completeness we will briefly review the nonlinear CSs, as Man'ko *et al* introduced in section 2, following with a review on KPS and GK coherent states in section 3. Then the relation between KPS, PS and BG coherent states of $su(1, 1)$ with nonlinear CSs will be obvious in section 4, and so the generators of the deformed oscillator algebra, displacement type operator and the proper Hamiltonian in each set of the above coherent states will be found in sections 5 and 6. Based on our results we will introduce a vast class of new generalized CSs (*dual* family of KPS and PS coherent states). Then we discuss the extension of the procedure to the GK coherent states in section 7, and finally we present our conclusions.

2. Nonlinear coherent states

Nonlinear CSs were first introduced explicitly in de Matos Filho and Vogel (1996) and Man'ko *et al* (1997), but before them they were implicitly defined by Shanta *et al* (1994) in a compact form. This notion attracted much attention in the physical literature in recent decades, especially because of their nonclassical properties in quantum optics. Man'ko *et al*'s approach is based on the two following postulates.

The first is that the standard annihilation and creation operators deformed with an intensity dependent function $f(\hat{n})$ (which is an operator valued function), according to the relations

$$A = af(\hat{n}) = f(\hat{n}+1)a \tag{1}$$

$$A^{\dagger} = f^{\dagger}(\hat{n})a^{\dagger} = a^{\dagger}f^{\dagger}(\hat{n}+1) \tag{2}$$

with commutators between A and $A^{\dagger}$ as

$$[A, A^{\dagger}] = (\hat{n}+1)f(\hat{n}+1)f^{\dagger}(\hat{n}+1) - \hat{n}f^{\dagger}(\hat{n})f(\hat{n}) \tag{3}$$

where a, $a^{\dagger}$ and $\hat{n} = a^{\dagger}a$ are bosonic annihilation, creation and number operators, respectively. Ordinarily the phase of f is irrelevant and one may choose f to be real and non-negative, i.e. $f^{\dagger}(\hat{n}) = f(\hat{n})$. But to keep general consideration, we take into account the phase dependence of $f(\hat{n})$ in general formalism given in this section.

The second postulate is that the Hamiltonian of the deformed oscillator in analogy with the harmonic oscillator is found to be

$$\hat{H}_M = \tfrac{1}{2}(AA^{\dagger} + A^{\dagger}A) \tag{4}$$

which by equations (1) and (2) can be rewritten as

$$\hat{H}_M = \tfrac{1}{2}((\hat{n}+1)f(\hat{n}+1)f^{\dagger}(\hat{n}+1) + \hat{n}f^{\dagger}(\hat{n})f(\hat{n})). \tag{5}$$

where by index M we want to denote the Hamiltonian as introduced by Man'ko *et al*. The single mode nonlinear CSs obtained as an eigenstate of the annihilation operator is as follows:

$$|z\rangle_{\mathrm{NL}} = \mathcal{N}_f(|z|^2)^{-1/2}\sum_{n=0}^{\infty} C_n z^n |n\rangle \tag{6}$$

where the coefficients C_n are given by

$$C_n = \left(\sqrt{[nf^{\dagger}(n)f(n)]!}\right)^{-1} \qquad C_0 = 1 \qquad [f(n)]! \doteq f(n)f(n-1)\ldots f(1) \tag{7}$$

and the normalization constant is determined as

$$\mathcal{N}_f(|z|^2) = \sum_{n=0}^{\infty} |C_n|^2 |z|^{2n}. \tag{8}$$

In order to have states belonging to the Fock space, it is required that $0 < \mathcal{N}_f(|z|^2) < \infty$, which implies that $|z| \leqslant \lim_{n\mapsto\infty} n[f(n)]^2$. No further restrictions are then put on $f(n)$. Now with the help of equations (6) and (7) the function $f(n)$ corresponding to any nonlinear CS is found to be

$$f(n) = \frac{C_{n-1}}{\sqrt{n}C_n} \tag{9}$$

which plays the key role in our present work. To recognize the nonlinearity of any CS we can use this simple and useful relation; by this we mean that if C_n for any CS are known, then $f(n)$ can be found from equation (9); when $f(n) = 1$ or at most is only a constant phase, we recover the original oscillator algebra, otherwise it is nonlinear.

3. KPS and GK generalized coherent states

Along with the generalization of CSs Klauder *et al* (2001) introduced the states

$$|z\rangle_{\mathrm{KPS}} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle \tag{10}$$

where $\rho(n)$ satisfies $\rho(0) = 1$ and the normalization constant is determined as

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho(n)}. \tag{11}$$

Comparing equations (6) with (10) we obtain $C(n) = [\rho(n)]^{-1/2}$ which describes the relation between KPS and nonlinear CSs. Indeed when $C_n = 1/\sqrt{n!}$ we obtain CCS, otherwise (for instance in general when $C_n = 1/\sqrt{\rho(n)}$) as in (10), we are led to nonlinear CSs, this is what we want to show. We will demonstrate in detail the equivalence between all sets of KPS coherent states and the nonlinear CSs in section 6. These states possess the three conditions (i)–(iii) stated in the introduction, by appropriately selected functions $\rho(n)$. The third condition, which is the most difficult and at the same time the strongest requirement of any set of CSs, was proved appreciatively by them, through Stieltjes and Hausdorff power-moment problem. Explicitly, for each set of CSs $|z\rangle_{\mathrm{KPS}}$, they found the positive weight function $W(|z|^2)$ such that

$$\int\int_{\mathbb{C}} \mathrm{d}^2 z |z\rangle_{\mathrm{KPS}} W(|z|^2)_{\mathrm{KPS}}\langle z| = \hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n| \tag{12}$$

where $\mathrm{d}^2 z = |z| d|z| \, \mathrm{d}\theta$. Strictly speaking, evaluating the integral over θ in the LHS of equation (12), setting $|z|^2 \equiv x$ and simplifying it, we arrive finally at

$$\int_0^R x^n \tilde{W}(x)\, \mathrm{d}x = \rho(n) \qquad n = 0, 1, 2, \ldots \qquad 0 < R \leqslant \infty \tag{13}$$

where the positive weight functions $\tilde{W}(x) = \frac{\pi W(x)}{\mathcal{N}(x)}$ must be determined. Extending the values of $n \in \mathbb{N}$ in (13) to $s \in \mathbb{C}$, it can be rewritten as

$$\int_0^\infty x^{s-1}\tilde{W}(x)\,\mathrm{d}x = \rho(s-1) \tag{14}$$

for $R = \infty$, which is known as the Stieltjes moment problem and

$$\int_0^\infty x^{s-1}H(R-x)\tilde{W}(x)\,\mathrm{d}x = \rho(s-1) \tag{15}$$

for $R < \infty$, which is known as the Hausdorff moment problem, with $H(R-x)$ as the Heaviside function. The positive weight functions $\tilde{W}(x)$ can be obtained, through Mellin and inverse Mellin transform techniques, when $\rho(n)$ is chosen. This automatically guarantees the condition of resolution of the identity for the KPS states according to (12) (for a brief and nice discussion see Klauder (2001) and references therein, e.g., Marichev (1983) and Prudinkov and Brychkov (1998)).

Adopting certain physical criteria rather than imposing selected mathematical requirements, Klauder and Gazeau by reparametrizing the generalized CSs $|z\rangle$ in terms of a two independent parameters J and γ, introduced the generalized CSs $|J,\gamma\rangle$, known ordinarily as Gazeau–Klauder (GK) coherent states in the physical literature (Klauder 1998, Gazeau *et al* 1999). These are explicitly defined by the expansion

$$|J,\gamma\rangle = \mathcal{N}(J)^{-1/2}\sum_{n=0}^{\infty}\frac{J^{n/2}\,\mathrm{e}^{-\mathrm{i}e_n\gamma}}{\sqrt{\rho(n)}}|n\rangle \tag{16}$$

where the normalization constant is given by

$$\mathcal{N}(J) = \sum_{n=0}^{\infty}\frac{J^n}{\rho(n)} \tag{17}$$

and $\rho(n)$ is a positive weight factor with $\rho(0) \equiv 1$ by convention and the domains of J and γ are such that $J \geqslant 0$ and $-\infty < \gamma < \infty$. These states required the following properties to be satisfied: (i) continuity of labelling: if $(J,\gamma) \to (J',\gamma')$ then, $\||J,\gamma\rangle - |J',\gamma'\rangle\| \to 0$, (ii) resolution of the identity: $\hat{I} = \int |J,\gamma\rangle\langle J,\gamma| d\mu(J,\gamma)$ as usual and two extra properties: (iii) temporal stability: $\exp(-\mathrm{i}\hat{H}t)|J,\gamma\rangle = |J,\gamma+\omega t\rangle$ and (iv) the action identity: $H = \langle J,\gamma|\hat{H}|J,\gamma\rangle = \omega J$, where H and $\hat{H}$ are classical and quantum mechanical Hamiltonians of the system, respectively. It must be understood that the fourth condition forced the generalized CSs to have the essential property: 'the most classical quantum states', but now in the sense of *energy* of the dynamical system, in the same way that the canonical coherent state (CCS) is a quantum state whose position and momentum expectation values obey the classical orbits of a harmonic oscillator in phase space.

In equation (16) the kets $|n\rangle$ are the eigenvectors of the Hamiltonian $\hat{H}$, with the eigen-energies E_n

$$\hat{H}|n\rangle = E_n|n\rangle \equiv \hbar\omega e_n|n\rangle \equiv e_n|n\rangle \qquad \hbar \equiv 1 \qquad \omega \equiv 1 \qquad n = 0, 1, 2, \ldots. \tag{18}$$

The action identity *uniquely* specified $\rho(n)$ in terms of the eigenvalues of the Hamiltonian $\hat{H}$ with a discrete spectrum $0 = e_0 < e_1 < e_2 < \ldots$

$$\rho(n) = \Pi_{k=1}^{n} e_k \equiv [e_n]!. \tag{19}$$

As an example, for the shifted Hamiltonian of a harmonic oscillator we have the CCS denoted by $|J,\gamma\rangle_{\mathrm{CCS}}$:

$$|J,\gamma\rangle_{\mathrm{CCS}} = \mathrm{e}^{-J/2}\sum_{n=0}^{\infty}\frac{J^{n/2}\,\mathrm{e}^{-\mathrm{i}n\gamma}}{\sqrt{n!}}|n\rangle. \tag{20}$$

Equation (19) obviously states that $\rho(n)$ is directly related to the spectrum of the dynamical system. So every Hamiltonian *uniquely* determined the associated CS, although the inverse is not true. This is because of the existence of the isospectral Hamiltonians in the context of quantum mechanics (Fern'andez 1994, Roknizadeh and Tavassoly 2004).

4. The relation between some classes of generalized coherent states and the nonlinear coherent states

To start with we demonstrate the relation between the KPS and NL coherent states. As we have mentioned already, by comparing equations (6) and (10), the coefficients C_n can be determined. Then inserting C_n and C_{n-1} into (9) yields

$$f_{\mathrm{KPS}}(\hat{n}) = \sqrt{\frac{\rho(\hat{n})}{\hat{n}\rho(\hat{n}-1)}} \tag{21}$$

which simply provides a bridge between KPS and NL coherent states. As a special case, when $\rho(n) = n!$, i.e. the canonical CS, we obtain $f(n) = 1$. We will demonstrate the equivalence between all sets of KPS coherent states and the nonlinear CSs in section 6. From equation (19) e_n can easily be found as a function of $\rho(n)$:

$$e_n = \frac{\rho(n)}{\rho(n-1)}. \tag{22}$$

Remembering that e_ns are the eigenvalues of the Hamiltonian, it will be obvious that neither every $\rho(n)$ of KPS coherent states nor every $f(n)$ of nonlinear CSs are physically acceptable, when the dynamics of the system (Hamiltonian) is specified. Using equations (21) and (22) we get

$$f_{\mathrm{KPS}}(n) = \sqrt{\frac{e_n}{n}}, \qquad e_n = n(f_{\mathrm{KPS}}(n))^2. \tag{23}$$

Now by using (21) we are able to find $f(\hat{n})$ for all sets of KPS coherent states and then the deformed annihilation and creation operators $A = af(\hat{n})$ and $A^\dagger = f^\dagger(\hat{n})a^\dagger$ may easily be obtained. Returning to the above descriptions, the operators A and $A^\dagger$ satisfy the following relations:

$$A|n\rangle = \sqrt{e_n}|n-1\rangle \tag{24}$$

$$A^\dagger|n\rangle = \sqrt{e_{n+1}}|n+1\rangle \tag{25}$$

$$[A, A^\dagger]|n\rangle = (e_{n+1} - e_n)|n\rangle, \qquad [A, \hat{n}] = A, \qquad [A^\dagger, \hat{n}] = -A^\dagger. \tag{26}$$

Also note that $A^\dagger A|n\rangle = e_n|n\rangle$, not equal to $n|n\rangle$ in general. With these results in mind it would be obvious that equation (23) is not consistent with relations (4) and (5) for the Hamiltonian. A closer look at equation (23) which is a consequence of imposing the action identity on the Hamiltonian of the system leads us to obtain a new form of the Hamiltonian for the nonlinear CSs as

$$\hat{H} = \hat{n}f^2(\hat{n}) = A^\dagger A. \tag{27}$$

After all this form of the Hamiltonian may be considered as *'normal-ordered'* of the Man'ko *et al* Hamiltonian H_M, introduced in equation (4),

$$\hat{H} =: \hat{H}_M := \tfrac{1}{2} : A^\dagger A + AA^\dagger : . \tag{28}$$

Therefore the associated Hamiltonian for the KPS coherent state can be written as

$$\hat{H}_{\mathrm{KPS}} = \hat{n}(f_{\mathrm{KPS}}(\hat{n}))^2 = \frac{\rho(\hat{n})}{\rho(\hat{n}-1)}. \tag{29}$$

Comparing equations (4) and (5) with equations (23), (27) and (28) implies that if we require that the KPS and nonlinear CSs to possess the action identity property, the associated Hamiltonian when expressed in terms of ladder operators must be reformed in normal-ordered form.

In summary our considerations enable one to obtain f-deformed annihilation and creation operators as well as the Hamiltonian for all sets of $|z\rangle_{\mathrm{KPS}}$ discussed in Klauder (2001), after demonstrating that for each of them there exists a special nonlinearity function $f(\hat{n})$. Before closing this section, we give two illustrative examples to show the ability of our method.

Example 4.1. Penson–Solomon generalized CSs

As an example we can simply deduce the nonlinearity function for the generalized CSs introduced by Penson and Solomon (PS) (1999):

$$|q, z\rangle_{\mathrm{PS}} = \mathcal{N}(q, |z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{\sqrt{n!}} z^n |n\rangle \tag{30}$$

where $\mathcal{N}(q, |z|^2)^{-1/2}$ is a normalization function and $\varepsilon(q, 0) = 1, 0 \leqslant q \leqslant 1$. This definition is based on an entirely analytical prescription, in which the authors proposed the generalized exponential function obtained from the following differential equation:

$$\frac{\mathrm{d}\varepsilon(q, z)}{\mathrm{d}z} = \varepsilon(q, qz) \Rightarrow \varepsilon(q, z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{n!} z^n. \tag{31}$$

So they have not used the ladder (or displacement) operator definition for their states (Penson 1999). The nonlinearity function, the annihilation and creation operators evolving in these states can be easily obtained by our formalism as

$$f_{\mathrm{PS}}(\hat{n}) = q^{1-\hat{n}} \qquad A = aq^{1-\hat{n}} \qquad A^\dagger = q^{1-\hat{n}} a^\dagger. \tag{32}$$

Therefore our method enables one to reproduce them through solving the eigenvalue equation: $A|z, q\rangle_{\mathrm{PS}} = aq^{1-\hat{n}}|z, q\rangle_{\mathrm{PS}} = z|z, q\rangle_{\mathrm{PS}}$, in addition to introducing an $\hat{n}$-dependent Hamiltonian describing the dynamics of the system

$$\hat{H}_{\mathrm{PS}} = \hat{n} q^{2(1-\hat{n})}. \tag{33}$$

Example 4.2. Barut–Girardello CSs for $su(1,1)$ Lie algebra

As second example we refer to the Barut–Girardello (BG) CSs, defined for the discrete series representations of the group $SU(1,1)$ (Barut and Girardello 1971). Our formalism provides an easier tool to relate the $SU(1,1)$ CSs of BG to both the Pöschl–Teller (PT) and infinite square well potentials. The BG states decomposed over the number-state bases as

$$|z, \kappa\rangle_{\mathrm{BG}} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(n+2\kappa)}} |n\rangle \tag{34}$$

where $\mathcal{N}(|z|^2)^{-1/2}$ is a normalization constant and the label κ takes only the values $1, 3/2, 2, 5/2, \ldots$. Using (9) for these states we find

$$f_{\mathrm{BG}}(\hat{n}) = \sqrt{\hat{n} + 2\kappa - 1} \qquad \hat{H}_{\mathrm{BG}} = \hat{n}(\hat{n} + 2\kappa - 1) \qquad n = 0, 1, 2, \ldots. \tag{35}$$

So we have the eigenvalue equation

$$\hat{H}_{\mathrm{BG}}(\hat{n})|n\rangle = n(n + 2\kappa - 1)|n\rangle \tag{36}$$

with $e_n = n(n + 2\kappa - 1)$; when $\kappa = 3/2$ and $\kappa = \lambda + \eta$ (λ and η are the two parameters that characterize the PT potential: $\kappa = [(\lambda + \eta + 1)/2] > 3/2$) we obtain the infinite square well and PT potentials, respectively (Antoine *et al* 2001). Therefore we have established that the dynamical group associated with these two potentials is the $SU(1, 1)$ group. Also if we take into account the action of $A = af_{\mathrm{BG}}(\hat{n})$, $A^\dagger = f_{\mathrm{BG}}(\hat{n})a^\dagger$ and $[A, A^\dagger]$ on the states $|\kappa, n\rangle$ we obtain

$$A|\kappa, n\rangle = \sqrt{n(n + 2\kappa - 1)}|\kappa, n - 1\rangle \tag{37}$$

$$A^\dagger|\kappa, n\rangle = \sqrt{(n + 2\kappa)(n + 1)}|\kappa, n + 1\rangle \tag{38}$$

$$[A, A^\dagger]|\kappa, n\rangle = (n + \kappa)|\kappa, n\rangle. \tag{39}$$

We conclude that the generators of $su(1, 1)$ algebra L_-, L_+, L_{12} can be expressed in terms of the deformed annihilation and creation operators including their commutators such that

$$\begin{aligned} L_- &\equiv \tfrac{1}{\sqrt{2}}A = \tfrac{1}{\sqrt{2}}af_{\mathrm{BG}}(\hat{n}) \\ L_+ &\equiv \tfrac{1}{\sqrt{2}}A^\dagger = \tfrac{1}{\sqrt{2}}f_{\mathrm{BG}}(\hat{n})a^\dagger \\ L_{12} &\equiv \tfrac{1}{2}[A, A^\dagger]. \end{aligned} \tag{40}$$

We note that our approach not only recover the results of Antoine (2001) in a simpler and clearer manner, but also it gives the explicit form of the operators L_-, L_+, L_{12} as some '*intensity dependent*' operators consistent with the Holstein–Primakoff single mode realization of the $su(1, 1)$ Lie algebra (Gerry 1983).

Example 4.3. $su(1, 1)$-Barut–Girardello CSs for Landau levels

As an example closer to physics, it is well known that the Landau levels (LL) are directly related to quantum mechanical study of the motion of a charged and spinless particle on a flat plane in a constant magnetic field (Landau and Lifshitz 1977, Gazeau *et al* 2002). Recently it was realized two distinct symmetries corresponded to these states, namely $su(2)$ and $su(1, 1)$ (Fakhri 2004). The author showed that the quantum states of the Landau problem corresponding to the motion of a spinless charged particle on a flat surface in a constant magnetic field $\beta/2$ along z-axis may be obtained as

$$|n, m\rangle = \frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}}\left(\frac{r}{2}\right)^{\frac{2\alpha+1}{2}} \mathrm{e}^{-\beta r^2/8} L_{n,m}^{(\alpha,\beta)}\left(\frac{r^2}{4}\right) \tag{41}$$

where $0 \leqslant \varphi \leqslant 2\pi$, $\alpha > -1$, $n \geqslant 0$, $0 \leqslant m \leqslant n$ and $L_{n,m}^{(\alpha,\beta)}$ are the associated Laguerre functions. Constructing the Hilbert space spanned by $\mathfrak{H} := \{|n, m\rangle\}_{n\geqslant 0, 0\leqslant m\leqslant n}$, there it is shown that the Barut–Girardello CSs (BGCSs)-type associated with this system can be obtained as the following combination of the orthonormal basis:

$$|z\rangle_m = \frac{|z|^{(\alpha+m)/2}}{\sqrt{I_{\alpha+m}(2|z|)}} \sum_{n=m}^{\infty} \frac{z^{n-m}}{\sqrt{\Gamma(n - m + 1)\Gamma(\alpha + n + 1)}}|n, m\rangle \tag{42}$$

where $I_{\alpha+m}(2|z|)$ is the modified Bessel function of the first kind (Watson 1995). The states in (42) are derived with the help of the lowering generator of the $su(1, 1)$ Lie algebra, the action of which is defined by the relation:

$$K_-|n, m\rangle = \sqrt{(n + \alpha)(n - m)}|n - 1, m\rangle \qquad K_-|m, m\rangle = 0. \tag{43}$$

A deep inspection of the states in (42) in comparison to the states in (34) shows little difference, in view of the lower limit of summation sign in the former equation. But this situation is similar to the states known as photon-added CSs (Agarwal and Tara 1991) in the sense that both of them are combinations of Fock space, with a cut-off in the summation from below. This common feature leads us to go on with the same procedure that has been done already to yield the nonlinearity function of the photon-added CSs (PACSs) in Wang and Fu (1999) and Sivakumar (1999). We define the deformed annihilation operator and the nonlinearity function similar to PACSs as

$$A = f(\hat{n})a, \qquad f(n) = \frac{C_n}{\sqrt{n+1}C_{n+1}} \tag{44}$$

where we have used the states (6) (replacing $|n\rangle$ by $|n, m\rangle$) as the eigenstates of the new annihilation operator defined in (44). Upon these considerations we can calculate the nonlinearity function for the $su(1, 1)$-BGCSs related to LL as follows:

$$f_{\mathrm{LL}}(\hat{n}) = \frac{(\hat{n}-m+1)(\hat{n}+\alpha+1)}{\sqrt{\hat{n}+1}}. \tag{45}$$

Hence again the ladder operators corresponding to this system may be obtained easily. Besides this, we will observe in the next section that one can also find displacement operators associated with the above three examples, as well as KPS and GK coherent states.

5. Introducing the generalized displacement operators

After we recast the KPS, PS and $su(1, 1)$ coherent states as nonlinear CSs and found a nonlinearity function for each set of them, it is now possible to construct all of them through a displacement type operator formalism. We will do this in two distinct ways.

(1) Roy and Roy (2000) gave a proposition and defined two new operators as follows:

$$B = a\frac{1}{f(\hat{n})} \qquad B^\dagger = \frac{1}{f(\hat{n})}a^\dagger. \tag{46}$$

Before we proceed to further clarify the problem, an interesting result may be given here. Choosing a special composition of the operators A in (1) and $B^\dagger$ in (46), we may observe that $B^\dagger A|n\rangle = n|n\rangle = A^\dagger B|n\rangle$. Also due to the following commutation relations:

$$[A, B^\dagger] = I, \qquad [A, B^\dagger A] = A, \qquad [B^\dagger, B^\dagger A] = -B^\dagger \tag{47}$$

the generators $\{A, B^\dagger, B^\dagger A, I\}$ constitute the commutation relations of the Lie algebra h_4. The corresponding Lie group is the well known Weyl–Heisenberg (W–H) group denoted by H_4. The same situation holds for the set of generators $\{B, A^\dagger, A^\dagger B, I\}$, using $A^\dagger$ in (1) and B in (46).

Coming back again to the Roy and Roy formalism, the relations in (46) allow one to define two generalized displacement operators

$$D'(z) = \exp(zA^\dagger - z^* B) \tag{48}$$

$$D(z) = \exp(zB^\dagger - z^* A). \tag{49}$$

Noting that $D'(z) = D(-z)^\dagger = [D(z)^{-1}]^\dagger$, it may be realized that the *dual* pairs obtained generally from the actions of (48) and (49) on the vacuum state are the orbits of a projective *nonunitary* representation of the W–H group (Ali *et al* 2004), which we named *displacement type* or *generalized displacement* operator. Therefore it is possible to construct two sets of CSs, the first is the old one introduced in Klauder (2001):

$$D(z)|0\rangle \equiv |z\rangle_{\mathrm{KPS}} \tag{50}$$

and the other one which is a new family of CSs, named '*dual states*' in Roy and Roy (2000) and Ali *et al* (2004) is

$$D'(z)|0\rangle \equiv |z\rangle_{\mathrm{KPS}}^{\mathrm{dual}} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n \sqrt{\rho(n)}}{n!} |n\rangle \tag{51}$$

where the normalization constant is determined as

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n} \rho(n)}{(n!)^2}. \tag{52}$$

Obviously the states $|z\rangle_{\mathrm{KPS}}^{\mathrm{dual}}$ in (51) are new ones, other than $|z\rangle_{\mathrm{KPS}}$. By the same procedures we have done in the previous sections it may be seen that the new states, $|z\rangle_{\mathrm{KPS}}^{\mathrm{dual}}$ can also be considered as NL coherent states with the nonlinearity function

$$f_{\mathrm{KPS}}^{\mathrm{dual}}(\hat{n}) = \sqrt{\frac{\hat{n}\rho(\hat{n}-1)}{\rho(\hat{n})}} \tag{53}$$

which is exactly the inverse of $f_{\mathrm{KPS}}(\hat{n})$, as one may expect. Also the Hamiltonian for the dual oscillator is found to be

$$\hat{H}_{\mathrm{KPS}}^{\mathrm{dual}} = \hat{n}\left(f_{\mathrm{KPS}}^{\mathrm{dual}}(\hat{n})\right)^2 = \hat{n}^2 \hat{H}_{\mathrm{KPS}}^{-1}. \tag{54}$$

(2) More recently a general mathematical physics formalism for constructing the dual states has been proposed by Ali *et al* (2004). Following the latter formalism one can define the operator

$$\hat{T} = \sum_{n=0}^{\infty} \sqrt{\frac{n!}{\rho(n)}} |n\rangle\langle n| \tag{55}$$

the action of which on canonical CS, $|z\rangle_{\mathrm{CCS}} = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$, yields the KPS coherent state

$$\hat{T}|z\rangle_{\mathrm{CCS}} = |z\rangle_{\mathrm{KPS}}. \tag{56}$$

The $\hat{T}$ operator we introduced in equation (55) is well defined and the inverse of it can be easily obtained as

$$\hat{T}^{-1} = \sum_{n=0}^{\infty} \sqrt{\frac{\rho(n)}{n!}} |n\rangle\langle n| \tag{57}$$

by which we may construct the new family of dual states: $\hat{T}^{-1}|z\rangle_{\mathrm{CCS}} \equiv |z\rangle_{\mathrm{KPS}}^{\mathrm{dual}}$. It is readily found that these states are just the states we have obtained in (51). Before applying the formalism to the KPS coherent states we give some examples.

It must be mentioned that investigating the resolution of the identity and discussing the nonclassical properties, such as squeezing of the quadratures, sub (super)-Poissonian statistics, amplitude squared squeezing, bunching (or antibunching) and the metric factor of these new states in (51) employing the same $\rho(n)$ proposed in Klauder (2001) remain for future works.

Example 5.1. Generalized displacement operator for PS coherent states

Upon using the results we derived in example 4.1 the dual of the PS states can be easily obtained using the Roy and Roy approach (2000):

$$|q, z\rangle_{\mathrm{PS}}^{\mathrm{dual}} = \mathcal{N}(q, |z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{\sqrt{n!}} z^n |n\rangle \tag{58}$$

where $\mathcal{N}(q, |z|^2)$ is some normalization constant, which may be determined. For this example the proposition in Ali *et al* (2004) works well. The $\hat{T}$-operator in this case reads

$$\hat{T} = \sum_{n=0}^{\infty} q^{n(n-1)/2} |n\rangle\langle n| \tag{59}$$

by which we may obtain

$$\hat{T}^{-1}|z\rangle_{\mathrm{CCS}} \equiv \left(\sum_{n=0}^{\infty} q^{-n(n-1)/2} |n\rangle\langle n| \right) |z\rangle_{\mathrm{CCS}} = |z, q\rangle_{\mathrm{PS}}^{\mathrm{dual}} \tag{60}$$

which are exactly the dual states we obtained in equation (58).

Example 5.2. Generalized displacement operators for BG and GP coherent states of $su(1,1)$ Lie algebra

As a well-known example we express the dual of BG coherent states in example 4.2. The duality of these states with the so-called Gilmore–Perelomov (GP) CSs has already been demonstrated in Ali *et al* (2004). The latter states were defined as

$$|z, \kappa\rangle_{\mathrm{GP}} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \sqrt{\frac{(n+2\kappa-1)!}{n!}} z^n |n\rangle, \tag{61}$$

where $\mathcal{N}(|z|^2)$ is a normalization constant. The nonlinearity function and the Hamiltonian may be written as $f_{\mathrm{GP}}(\hat{n}) = f_{\mathrm{BG}}^{-1}(\hat{n})$ and $\hat{H} = \hat{n}/(\hat{n}+2\kappa-1)$. Therefore we have in this case

$$B|\kappa, n\rangle = \sqrt{\frac{n}{n+2\kappa-1}} |\kappa, n-1\rangle \tag{62}$$

$$B^{\dagger}|\kappa, n\rangle = \sqrt{\frac{n+1}{n+2\kappa}} |\kappa, n+1\rangle \tag{63}$$

$$[B, B^{\dagger}]|\kappa, n\rangle = \frac{2\kappa-1}{(n+2\kappa)(n+2\kappa-1)} |\kappa, n\rangle, \tag{64}$$

where $B = af_{\mathrm{GP}}(\hat{n})$ and $B^{\dagger} = f_{\mathrm{GP}}(\hat{n})a^{\dagger}$ are the deformed annihilation and creation operators for the dual system (GP states), respectively. It is immediately observed that $[A, B^{\dagger}] = I$, $[B, A^{\dagger}] = I$, where obviously $A = af_{\mathrm{BG}}(\hat{n})$ and $A^{\dagger}$ is its Hermitian conjugate. So it is possible to obtain the displacement type operators for the BG and GP nonlinear CSs discussed in this paper, using relations (48) and (49). As a result the displacement operators obtained by our method are such that

$$|z, \kappa\rangle_{\mathrm{BG}} = D_{\mathrm{BG}}(z)|\kappa, 0\rangle = \exp(zA^{\dagger} - z^{*}B)|\kappa, 0\rangle \tag{65}$$

and

$$|z, \kappa\rangle_{\mathrm{GP}} = D_{\mathrm{GP}}(z)|\kappa, 0\rangle = \exp(zB^{\dagger} - z^{*}A)|\kappa, 0\rangle. \tag{66}$$

To the best of our knowledge these forms of displacement type operators for discrete series representation of $SU(1,1)$ group related to BG and GP coherent states have not appeared in the literature up to now.

We close this section by observing that to apply the procedure to the $su(1,1)$-BG coherent states for LL we expressed as an example (example 4.3), one must redefine the auxiliary operators B and $B^{\dagger}$, in place of those introduced in (46), as follows:

$$B = \frac{1}{f_{\rm LL}(\hat{n})}a \qquad B^{\dagger} = a^{\dagger}\frac{1}{f_{\rm LL}(\hat{n})}. \tag{67}$$

The other calculations are the same as the general $su(1,1)$ coherent states in this example, so let us leave it here.

6. Constructing KPS coherent states through annihilation and displacement operators and introducing proper Hamiltonians for them

Now we list the nonlinearity function $f(\hat{n})$ and the associated Hamiltonian $\hat{H}(\hat{n})$ for all of the KPS coherent states according to our proposed method. The operators $A, A^{\dagger}$ can easily be obtained using equations (1), (2) and the displacement operators in a like manner may be obtained using equations (46), (48) and (49). So we ignore introducing them explicitly. A similar argument can also be made for the dual family of CSs (see (53) and (54) and the discussion before it for $f^{\rm dual}_{\rm KPS}(\hat{n})$ and $\hat{H}^{\rm dual}_{\rm KPS}(\hat{n})$). Therefore introducing $f(\hat{n})$ and $\hat{H}(\hat{n})$ seems enough for our purpose. The set of CSs defined on the whole plane is as follows.

(a) $\rho(n) = \frac{(n+p)!}{p!}$

$$f(\hat{n}) = \sqrt{\frac{\hat{n}+p}{\hat{n}}} \qquad \hat{H} = \hat{n} + p. \tag{68}$$

One can see from $\hat{H}$ that it is simply a shift in the energy eigenstates of the harmonic oscillator, as expected from $\rho(n)$, which is only a shift in $n!$. In all of the following cases we use the definitions (7) for $f(n)!$, whenever necessary.

(b) $\rho(n) = \frac{\Gamma(\alpha n+\beta)}{\Gamma(\beta)}$, where in this case and what follows Γ is the gamma function. The CSs constructed by this $\rho(n)$ are known as the Mittag–Leffler (ML) CSs (Sixdeniers *et al* 1999). As a special case for $\alpha = 1$ and β arbitrary we obtain

$$f(\hat{n}) = \sqrt{\frac{\hat{n}+\beta-1}{\hat{n}}} \qquad \hat{H} = \hat{n} + \beta - 1 \tag{69}$$

where we have used the recurrence relation $\Gamma(z+1) = z\Gamma(z)$. As a result we observe that the ML coherent states are also nonlinear CSs.

(c) $\rho(n) = \frac{n!}{n+1}$

$$f(\hat{n}) = \sqrt{\frac{\hat{n}}{\hat{n}+1}} \qquad \hat{H} = \frac{\hat{n}^2}{\hat{n}+1}. \tag{70}$$

The dual of this state occurs for $\rho(n) = (n+1)!$, which is a special case of the item (a) when $p = 1$. So there is no problem with the resolution of the identity for them.

(d) $\rho(n) = \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)(1+n)}$

$$f(\hat{n}) = \sqrt{\frac{\hat{n}+\alpha}{\hat{n}+1}} \qquad \hat{H} = \frac{\hat{n}(\hat{n}+\alpha)}{\hat{n}+1}. \tag{71}$$

(e) $\rho(n) = (n!)^2$

$$f(\hat{n}) = \sqrt{\hat{n}} \qquad \hat{H} = \hat{n}^2. \tag{72}$$

This kind of deformation has already been employed by Buzek (1989) who imposed on the single-mode field operators a and $a^\dagger$, and showed that in the Jaynes–Cummings model with intensity-dependent coupling (when the notion of nonlinear CS had still not been used up to that time), interacting with the Holstein–Primakoff $SU(1,1)$ CSs, the revivals of the radiation squeezing are strictly periodical for any value of initial squeezing. Also the dual of these states with $f(\hat{n}) = 1/\sqrt{\hat{n}}$ has already been discovered and named *harmonious states* (Sudarshan 1993).

(f) $\rho(n) = (n!)^3$

$$f(\hat{n}) = \hat{n} \qquad \hat{H} = \hat{n}^3. \tag{73}$$

(g) $\rho(n) = \frac{n!\Gamma(n+3/4)}{\Gamma(4/3)}$

$$f(\hat{n}) = \sqrt{\hat{n}+1/3} \qquad \hat{H} = \hat{n}(\hat{n}+1/3). \tag{74}$$

(h) $\rho(n) = \frac{(n!)^{3/2}\Gamma(3/2)}{\Gamma(n+3/2)}$

$$f(\hat{n}) = \frac{\hat{n}}{\sqrt{\hat{n}+1/2}} \qquad \hat{H} = \frac{\hat{n}^3}{\hat{n}+1/2}. \tag{75}$$

All the Hamiltonians we derived above are such that $\lim_{n\to\infty} E_n = \infty$. Upon further investigation of the above f-functions, an interesting physical point which may be explored is that the $f(n)$ functions in (68), (69), (70) and (71) will be equal to 1 (canonical CS) in the limit $n \to \infty$, i.e. for high intensities. So to observe the nonlinearity phenomenon and its features in these special cases we do not need high intensities of light. In contrast, if anyone can generate these special states through some physical processes, for instance, field–atom interaction in a cavity, the nonlinearity effects can be detected at *low intensities*.

In addition to the above sets of CSs, Klauder *et al* introduced a large class of CSs which has been defined on a unit disc. For these states also it is possible to continue in the same way as we did for the states on the whole of the complex plane:

(a′) $\rho(n) = \frac{2}{n+2}$

$$f(\hat{n}) = \sqrt{\frac{\hat{n}+1}{\hat{n}(\hat{n}+2)}} \qquad \hat{H} = \frac{\hat{n}+1}{\hat{n}+2}. \tag{76}$$

(b′) $\rho(n) = \frac{6}{(n+2)(n+3)}$

$$f(\hat{n}) = \sqrt{\frac{\hat{n}+1}{\hat{n}(\hat{n}+3)}} \qquad \hat{H} = \frac{\hat{n}+1}{\hat{n}+3}. \tag{77}$$

(c′) $\rho(n) = \frac{\pi}{4}\frac{(n!)^2}{\Gamma^2(n+3/2)}$

$$f(\hat{n}) = \frac{2\sqrt{\hat{n}}}{2\hat{n}+1} \qquad \hat{H} = \frac{4\hat{n}^2}{(2\hat{n}+1)^2}. \tag{78}$$

(d′) $\rho(n) = \frac{3\pi}{8}\frac{n!(n+1)!}{\Gamma(n+3/2)\Gamma(n+5/2)}$

$$f(\hat{n}) = 2\sqrt{\frac{\hat{n}+1}{(2\hat{n}+1)(2\hat{n}+3)}} \qquad \hat{H} = \frac{4\hat{n}(\hat{n}+1)}{(2\hat{n}+1)(2\hat{n}+3)}. \tag{79}$$

(e′) $\rho(n) = \frac{\Gamma(1+c-a)\Gamma(1+c-b)\Gamma(n+1)\Gamma(n+1-a-b)}{\Gamma(1+c-a-b)\Gamma(n+1+c-a)\Gamma(n+1+c-b)}$, setting $a = b = 1/2$ and $c = 3/2$ we have

$$f(\hat{n}) = \frac{\sqrt{\hat{n}+1/2}}{\hat{n}+1} \qquad \hat{H} = \frac{\hat{n}(\hat{n}+1/2)}{(\hat{n}+1)^2}. \tag{80}$$

(f′) $\rho(n) = \frac{3\Gamma(5/2)(n+1)!}{(n+3)\Gamma(n+5/2)}$

$$f(\hat{n}) = \sqrt{\frac{\hat{n}^2+3\hat{n}+2}{\hat{n}(\hat{n}+3)(\hat{n}+3/2)}} \qquad \hat{H} = \frac{\hat{n}^2+3\hat{n}+2}{(\hat{n}+3)(\hat{n}+3/2)}. \tag{81}$$

The common property of these states is that the limits of $f(n)$ and H as n goes to infinity are 0 and 1, respectively.

One can easily check that using the above f-functions (defined on the whole plane or restricted to the open unit disc) and solving the eigenvalue equation $A|z\rangle = z|z\rangle$, or acting with the displacement operator $D(z)$, obtained from equation (50) on the vacuum state $|0\rangle$ by the well-known procedures, correctly leads to the $|z\rangle_{\mathrm{KPS}}$. The same argument may be followed for the dual family $|z\rangle^{\mathrm{dual}}_{\mathrm{KPS}}$.

7. A discussion on constructing GK coherent states by annihilation and displacement operators techniques

Klauder asked the question, "What is the physics involved in choosing the annihilation operator eigenstates"? Therefore he and Gazeau redefined the generalized CSs with the four requirements (i)–(iv) mentioned before in section 3 (Klauder 1998, 2001, Gazeau and Klauder 1999), nowadays known as GK coherent states (equation (16)). To discover the nonlinearity nature of the GK coherent states one may apply the method we proposed, on these states. As a result we arrive at the following expression for $f_{GK}(\gamma, \hat{n})$:

$$f_{GK}(\gamma, \hat{n}) = \mathrm{e}^{\mathrm{i}\gamma(\hat{e}_n - \hat{e}_{n-1})} = \sqrt{\frac{\rho(\hat{n})}{\hat{n}\rho(\hat{n}-1)}} \tag{82}$$

where we have chosen the notation $\hat{e}_n \equiv \frac{\rho(\hat{n})}{\rho(\hat{n}-1)}$ for simplicity. We demonstrated in the previous sections that this is the starting point to define the GK coherent states of any physical system as the (deformed) annihilation operator eigenstates and associate with them a displacement (type) operator. In addition to deducing the above results in detail, we applied the procedure to the known examples in the literature such as the Coulomb-like spectrum (Gazeau and Klauder 1999), the Pöschl–Teller and the infinite square well potentials (Antoine *et al* 2001), formally.

But, although we have introduced an expression for the nonlinearity function for the GK coherent states in equation (82), by which we can formally proceed further and get the explicit form of the requested results, upon a closer look at this relationship one can $f_{GK}(\gamma, \hat{n})$, is not a *well-defined operator valued function*, in view of highly careful mathematical considerations. It should be understood that clearly this result is not due to illegality of the proposed approach. In fact the difficulty in this special case might be expected naturally, because of the relaxing of the *holomorphicity* (Odzijewicz 1998) requirement in the definition of the GK coherent states (Gazeau and Klauder 1999).

Nevertheless one may still use the presented formalism to find some special dual family associated with GK coherent states, the explicit form of which will not be introduced here, until we find a well-defined form of them in future. Therefore the difficulties already mentioned in (Ali *et al* 2004) with the dual family of GK coherent states remain unsolved. We shall pay more attention to this matter in a forthcoming paper.

8. Concluding remarks

In summary we can classify the obtained results as follows.

(1) We connected some important classes of CSs: KPS, PS and GK coherent states and the discrete series representation of $su(1,1)$ Lie algebra using NL coherent states method successfully. Nonlinear CSs encompass all these states as special cases which are distinguishable from each other via the nonlinearity function $f(n)$. So we have obtained a '*unified method*' to construct all of these CSs which have already been introduced by different prescriptions. Therefore this work may be considered in parallel with previous efforts (Shanta *et al* 1994, Ali *et al* 2004) for unification of a large class of generalized CSs. Our re-construction of these states by the standard definitions, i.e. annihilation operator eigenstates and displacement operator techniques, with the preceding results enriching each set of the above classes of CSs in quantum optics, in the context of each other. So we enlarged the classification of nonlinear CSs with states having the main property of the generalized CSs (the resolution of the identity) considerably.

(2) We did not discuss the dual states in detail; indeed we wanted only to show that our method can use the Roy and Roy approach (2000) to produce the dual of any generalized CS that can be classified in the nonlinear CSs. So introduction of the dual family of generalized CSs coherent states has been done briefly, since it was out of the scope of our present work.

(3) The result we obtained in equation (27) is in fact the factorization implied in the literature, while by authors deal with solvable potentials and try to find ladder operators (see, for instance, Daoud (2002) and El Kinani and Daoud (2003)). According to the latter approach the one-dimensional supersymmetric quantum mechanics (SUSQM) provides an algebraic tool to define ladder operators for some exactly solvable potentials. Therefore it may be understood that our method can be considered as an alternative formalism parallel to the well-known SUSQM techniques to achieve this purpose more easily. Precisely speaking, for Hamiltonians having the properties mentioned in (18), i.e. for some special solvable systems, one can factorize the Hamiltonian as we did in (27). It is interesting that these two distinct approaches terminate at a common point. To clarify, we add here that the equations (8)–(11) in Daoud (2002), which describe the *action* of creation, annihilation and their commutators on the related Fock space (where the author did not introduce the explicit form of the related operators), can also be derived with the help of our method, at least formally (see the discussion in section 7). But since we are looking for not only their actions but also the explicit form of the related operators, in terms of standard ladder operators and number operator, we did not include them in this content due to the ill-definition of the nonlinearity function (82).

(4) As a postscript we mention another point. On the basis of the latter potentiality of such a naive approach, we proposed to find the creation and annihilation operators; it may be realized as a comment that our formalism provides a straightforward framework for producing the *photon-added* (Agarwal 1991) or more precisely *excited* CSs corresponding to exactly solvable potentials, such as Pöschl–Teller and Morse potentials (Popov 2003 and Daoud 2002). The procedure is really manifest. Once the eigenvalues of the Hamiltonian were known, the Gazeau–Klauder CSs can be constructed using equation (16). The next step is to find the nonlinearity function through which one can perform ladder operators with the help of our procedure. Finally the iterative action of the creation operator on the GK coherent states readily gives the excited CSs. In this way this procedure can be applied in general to all sets of CSs discussed in the present work

to create excited CSs of KPS, PS and $su(1,1)$ types, as well as the GK coherent states in a simple manner.

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